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The radial infall of a highly relativistic point particle into a Kerr black hole along the symmetry axis

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Abstract

In this Letter we consider the radial infall along the symmetry axis of an ultra-relativistic point particle into a rotating Kerr black hole. We use the Sasaki-Nakamura formalism to compute the waveform, energy spectra and total energy radiated during this process. We discuss possible connections between these results and the black hole-black hole collision at the speed of light process.

1 Introduction

The lack of exact radiative solutions to Einstein's equations has promoted perturbation theory in General Relativity into a special place, as the tool for analyzing gravitational radiation emitted in physically interesting events. The use of perturbation theory in spacetimes containing black holes started with the work of Regge and Wheeler [1], where they addressed the stability of the Schwarzschild geometry against small deviations. This analysis was extended to include the infall of particles in a Schwarzschild black hole by Zerilli [2] and others (see, e.g., [3]).

The Kerr geometry, without spherical symmetry, proved to be more difficult to handle, but has also given some insights into relations that previously seemed a mystery [4]. Teukolsky [5] was able to decouple and separate

the perturbation equations for the Newman-Penrose quantities, and reduced them to a single radial equation, now known as the Teukolsky equation. However, it suffers from a couple of drawbacks, in that it has a long range potential and a source term which is in general divergent at large distances. This makes it difficult to analyze radiation emitted by test particles in generic orbits, though special cases have been handled (see for example the case of circular orbits [6]). Sasaki and Nakamura [7] improved this situation by introducing a set of transformations that bring the Teukolsky equation into the Sasaki-Nakamura equation, which has both a short range potential and an everywhere well behaved source term. Moreover, their equation reduces to the Regge-Wheeler equation when the rotation parameter is set equal to zero. Using their formalism, Sasaki, Nakamura and co-workers [8] have computed the gravitational radiation for generic orbits of particles falling, from rest at infinity, into a Kerr black hole. These studies have recently been revisited [9, 10] not only to study the effects of radiation reaction, but also to produce accurate waveforms, to serve as templates to the various gravitational wave detectors already at work. An extension of the Sasaki-Nakamura formalism to perturbations other than gravitational is given in [11].

It is important to stress that this approach is only justified as long as μ , the mass of the infalling particle, is much smaller than M , the mass of the black hole, so that it can serve as perturbation parameter. By the late 70's however, when the first full numerical simulations of black hole collisions were being done [12], it became clear that taking the limit $\mu \rightarrow M$ gave unexpectedly good results in perturbation theory. For recent improvements see [13].

The high velocity collision of two black holes, a problem interesting in itself, has gained renewed interest recently, with the possibility of black hole formation at TeV scales [14]. This process was studied extensively some time ago by D'Eath and Payne [15], by performing a perturbation expansion around the Aichelburg-Sexl metric, which describes a Schwarzschild black hole moving at the speed of light. The question arises as to whether it is possible to treat this process in the old Regge-Wheeler-Zerilli approach. Is it possible to study the collision of two black holes moving at the speed of light, by considering some expansion around a Schwarzschild metric? In a previous paper [16] we have argued that it is possible, provided the agreement between perturbation theory and numerical relativity is more than a coincidence. In spite of this unproven universal agreement, we do have the strong conviction

that it will hold in this process, but we are ultimately justified by the excellent agreement between our results [16] and previous studies [15]. The essence of our previous study was simple: consider a highly relativistic particle falling into a Schwarzschild black hole, use the Zerilli approach to compute the waveform and energy, Lorentz-boost the black hole to high velocities in the direction of the infalling particle, and end up with a collision at high velocity between a black hole and a small test particle. If one employs the assumption that this still works if $\mu \rightarrow M$ the result follows. Our results [16] showed an excellent agreement with results by D'Eath and Payne [15] and with results by Smarr [17], which leads us to believe that once again perturbation theory gives very good results throughout all the values of the perturbation parameter.

One would now like to extend these results to the Kerr family of rotating black holes, a less studied geometry. D'Eath and Payne's results do not apply here for example, although with probably a great effort one could use their methods to study the collision of Aichelburg-Sexl-Kerr particles, which solution has been found in [18]. It is true that we do not expect to see black holes colliding at the speed of light in any astrophysical scenario, but we do expect to see this at Planck energies, or, perhaps at the LHC, should the TeV scenario [14] be correct. In any case, the holes will most probably be rotating. Furthermore, the work done so far in the Kerr geometry dealt only with infall which starts from rest, so it will be interesting to see the outcome when the particle has a non zero velocity at infinity, and compare it with results in the Schwarzschild geometry.

Thus, the situation we consider in this paper is the following: a highly relativistic point particle impinges radially into a Kerr black hole, along its symmetry axis. We will use the Sasaki-Nakamura formalism to find the energy radiated, which will be the main result of this paper. We will then Lorentz-boost the Kerr hole to high velocity in the direction of the infalling object, which basically amounts to put $M \rightarrow \gamma M$, where γ is the Lorentz factor. This will then describe the high velocity collision between a Kerr hole and a small particle. Finally, we put $\mu \rightarrow M$, which means that we are dealing with the collision at nearly the speed of light of two equal mass black holes, one rotating, the other non-rotating. The justification for doing this last step (not allowed on formal basis) comes from the excellent results obtained so far by perturbation theory [3, 12, 13, 16].

2 A Green's function solution to the Sasaki-Nakamura equation

After some manipulations, the Teukolsky equation [5] may be brought to the Sasaki-Nakamura [7] form (details about the Teukolsky formalism may be found in the original literature [5], and also in [19]. For a good account of the Sasaki-Nakamura formalism we refer the reader to [7],[8], and [9]):

$$\frac{d^2}{dr_*^2}X(\omega, r) - \mathcal{F}\frac{d}{dr_*}X(\omega, r) - \mathcal{U}X(\omega, r) = \mathcal{L}. \quad (1)$$

The tortoise r_* coordinate is defined by $dr_*/dr = (r^2 + a^2)/\Delta$, and ranges from $-\infty$ at the horizon to $+\infty$ at spatial infinity. The functions \mathcal{F} and \mathcal{U} can be found in the original literature [7, 8]. We are considering the radial infall of a highly relativistic particle into a Kerr black hole along the symmetry axis, so the situation is axisymmetric. This means that the azimuthal quantum number m appearing in \mathcal{F} and \mathcal{U} [8], may be set to zero. Also, in this simple situation it is possible to find an exact form for the function \mathcal{L} (compare this with the source term in Schwarzschild [16]):

$$\mathcal{L} = -\frac{\mu C \epsilon_0 \gamma_0 \Delta}{2\omega^2 r^2 (r^2 + a^2)^{3/2}} e^{-i\omega r_*}. \quad (2)$$

Here, $C = \left[\frac{8Z_l^{a\omega}}{\sin^2 \theta} \right]_{\theta=0}$ and $Z_l^{a\omega}(\theta)$ is a spin-weighted spheroidal harmonic [20]. The function $\gamma_0 = \gamma_0(r)$ can be found in [8]. The parameter ϵ_0 is the energy per unit rest mass of the infalling particle, which we take to be a very large quantity ($\epsilon_0 \rightarrow \infty$), since we are interested in highly relativistic particles. The Sasaki-Nakamura equation (1) is to be solved under the “only outgoing radiation at infinity” boundary condition, meaning

$$X(\omega, r) = X^{\text{out}} e^{i\omega r_*}, r_* \rightarrow \infty. \quad (3)$$

Once X^{out} is known, Teukolsky's radial function R can be found, when $r_* \rightarrow \infty$ (the region of interest here) as

$$R = -\frac{4\omega^2 X^{\text{out}}}{\lambda(\lambda + 2) - 12i\omega - 12a^2\omega^2} r^3 e^{i\omega r_*} = R^{\text{out}} r^3 e^{i\omega r_*}. \quad (4)$$

Following Nakamura and Sasaki [8] we define the multipolar structure so as to have

$$\Delta E = \frac{\pi}{4} \int_0^\infty \sum_{lm} |h^{lm}(\omega)|^2 + |h^{lm}(-\omega)|^2. \quad (5)$$

Once X is known, we can get the Teukolsky wavefunction near infinity from (4), and the energy radiated away from (5).

We would now like to find $X(\omega, r)$ from the Sasaki-Nakamura differential equation (1). This is accomplished by a Green's function technique, constructed so as to satisfy the usual boundary conditions, i.e., only ingoing waves at the horizon ($X \sim e^{-i\omega r_*}, r_* \rightarrow -\infty$) and outgoing waves at infinity ($X \sim e^{i\omega r_*}, r_* \rightarrow \infty$). We get that, near infinity (we are interested in knowing the wavefunction in this region),

$$X^{\text{out}} = -\frac{\mu\epsilon_0 c_0 C}{4i\omega^3 B} \int \frac{e^{-i\omega r_*} X^H}{r^2(r^2 + a^2)^{1/2}} dr. \quad (6)$$

Here $c_0 \equiv \gamma_0(r = \infty)$, and X^H is an homogeneous solution of (1) which asymptotically behaves as

$$X^H \sim A(\omega)e^{i\omega r_*} + B(\omega)e^{-i\omega r_*}, r_* \rightarrow \infty \quad (7)$$

$$X^H \sim e^{-i\omega r_*}, r_* \rightarrow -\infty. \quad (8)$$

We now discuss the numerical results.

3 Numerical Results and Conclusions

The main result of this paper is shown in Fig. 1, the energy spectra as a function of the angular quantum number l , for an “extreme” black hole with $a = 0.999M$. The results for other rotation parameters a are not shown, both because there are no qualitative changes and mostly because they follow from the discussion below. The most interesting and important features of these results are:

(i) Fig. 1 clearly shows that the zero frequency limit (ZFL) of the energy spectra, $\frac{dE}{d\omega} \omega \rightarrow 0$ is non-vanishing. We know [21] that the existence of a ZFL is closely linked to a non-zero velocity at infinity, so this comes as no surprise and was already observed [16] in the collision of a Schwarzschild black hole with an highly relativistic particle. The peculiar property of the ZFL arises

when we look more closely into the numerical results: the numerical data shows that the ZFL is exactly (up to the numerical error) equal to the ZFL for the infall of an highly relativistic particle into a Schwarzschild black hole. This means that the ZFL seems to be independent of the spin of the colliding objects, and given by Smarr's [17] expression.

(ii) The spectra is almost flat up to a critical (l -dependent) frequency, a fact also evident from Fig. 1. Again, this was also true for a non-rotating black hole [16, 17]. The spectra is flat up to $\omega \sim \omega_{QN}$, where ω_{QN} is the lowest quasinormal frequency for the Kerr black hole (for work on quasinormal modes on the Kerr geometry see [22], for example). For $\omega > \omega_{QN}$, the spectra decays exponentially, according to the empirical law $\frac{dE}{d\omega} \sim e^{-b\omega}$, ($\omega > \omega_{QN}$), first discovered by Davis et al [3], and further discussed by Sasaki and Nakamura [8].

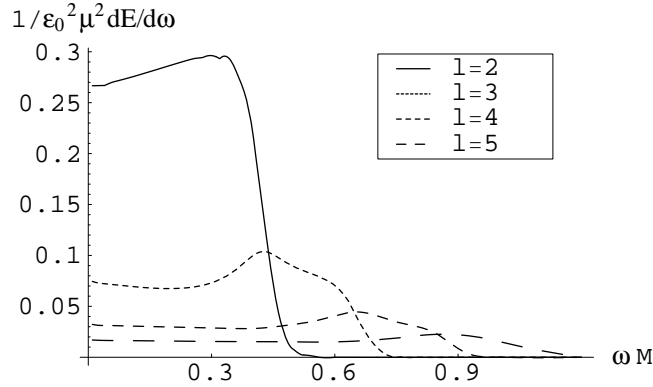


Figure 1. The energy spectra for a point particle moving at nearly the speed of light and colliding, along the symmetry axis, with an extreme ($a=0.999M$) Kerr black hole. Notice that the spectra is almost flat, the ZFL is non-vanishing and that the quadrupole carries less than half of the total radiated energy.

(iii) The total energy radiated in each multipole goes as $\Delta E_l \sim \frac{1}{l^2}$, which (needless to say) is exactly what happens for Schwarzschild case [16]. This power-law dependence seems to be *universal* for highly relativistic collisions, and was first observed for the non-rotating case [16]. We note that for collisions beginning at rest, the behaviour is strikingly different, for the energy

goes as $\Delta E_l(\text{rest at } \infty) \sim e^{-bl}$ [3, 8].

(iv) This power-law dependence implies that the radiation is not quadrupole ($l = 2$) in nature, as it was for infall starting from rest. In fact, less than 50% of the total energy is carried in the quadrupole part.

These characteristics plus the discussion in [16] allow one to infer that the total energy radiated will depend mainly on the behaviour of the quasinormal frequencies for the Kerr geometry. Now, for $m = 0$, we know [22] that the quasinormal frequencies for this geometry are almost the same (but slightly larger) as for the Schwarzschild geometry, so we expect the total energy to be similar in both cases (slightly larger in the Kerr case). In fact, for this axisymmetric collision we find, on summing over all l 's that

$$\Delta E = 0.31 \frac{\mu^2 \epsilon_0^2}{M} \quad a = 0.999M, \quad (9)$$

with a 5% error. We recall that for a Schwarzschild black hole [16] $\Delta E = 0.26 \frac{\mu^2 \epsilon_0^2}{M}$ so the infall along the symmetry axis of a Kerr hole does not enhance very much the total radiated energy (in comparison with the non-rotating case). This was also observed [8] for collisions along the symmetry axis, but starting from rest. As we lower the rotation parameter, the total energy decreases, and approaches the non-rotating value given above. What can we say about the collision at nearly the speed of light between a Schwarzschild and a Kerr black hole, along its symmetry axis? Supposing (with all due precautions mentioned in the Introduction) that (9) holds for $\mu \epsilon_0 \rightarrow M$, we should have an efficiency of 15, 5% for that process. This seems a reasonable result, but only a full numerical scheme can tell how accurate it is. For the moment, one can only say that it is a “good” result: it is higher than for the collision of two Schwarzschild black holes, but still within the upper limit imposed by the Area Theorem $\Delta E \leq 0.66 M$ and an efficiency less than 33% [23]. These numbers arise noting that the situation is axisymmetric, the radiation carries no angular momentum, and thus one has the same $a = 0.999M$ for the final black hole.

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